Online Appendix to "Inflation and Growth: A Non-Monotonic Relationship in an Innovation-Driven Economy"

Zhijie Zheng*
Southwestern University of Finance and Economics

Chien-Yu Huang†
Nanjing Audit University

Yibai Yang‡
University of Macau

January 28, 2019

Online Appendix A : Proofs of propositions

A.1 Proof of Propositions 1, 2 and 3

To analytically prove these propositions, first, we follow Segerstrom (2000) to establish the mutual R&D condition. This condition is derived from the first-order conditions of R&D profit maximizing problem, (14) and (19), for vertical and horizontal R&D firms. Substituting (11) into (14) yields the steady-state expected profit for each successful vertical innovative firm such that

$$\Pi_{vt} = \int_t^\infty e^{-\int_t^\tau (r+\phi_s)ds} \hat{\pi}_{t\tau} d\tau = \frac{\alpha(1-\alpha) L_y A_t^{\frac{1}{1-\alpha}}}{\rho + g_L + (\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}) g_A}. \tag{A.1}$$

Hence the two R&D conditions are written as

$$\frac{\delta \Gamma \alpha \lambda_v l_y \iota}{\rho + g_L + (\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}) g_A} l_v^{\delta-1} = 1 + \xi_v i, \tag{A.2}$$

and

$$\frac{\gamma \alpha \lambda_h l_y \iota}{\rho + g_L + (\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}) g_A} l_h^{\gamma-1} = 1 + \xi_h i. \tag{A.3}$$

*Research Institute of Economics and Management, Southwestern University of Finance and Economics, Chengdu, Sichuan, 611130, China. *Email* address: zhengzhijie1919@gmail.com.

†Institute of Urban Development, Nanjing Audit University, Nanjing, Jiangsu, 211815, China. *Email* address: chuang4@ncsu.edu.

‡Department of Economics, University of Macau, Taipa, Macao, China. *Email* address: yibai.yang@hotmail.com.

Combining (A.2) and (A.3) yields

$$\frac{\delta\lambda_v\Gamma l_v^{\delta-1}}{1+\xi_v i} = \frac{\gamma\lambda_h l_h^{\gamma-1}}{1+\xi_h i}. \tag{A.4}$$

Furthermore, using (27) and (28), (A.4) can be re-expressed as a relationship with two innovation growth rates, which is the *mutual R&D condition*, given by

$$g_N = \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}} g_A, \tag{A.5}$$

where $\Omega = \frac{1+\xi_h i}{1+\xi_v i}\Psi$ and $\Psi = \frac{\delta\Gamma\lambda_v}{\gamma\lambda_h}$. Substituting (24), (26) and $c_t = C_t/L_t$ into the individual's consumption-leisure condition (5) yields

$$l = 1 - \theta(1+\alpha)(1+\xi_c i)l_y. \tag{A.6}$$

Using (A.4), (A.6) and the labor market-clearing condition $l_y + l_v + l_h = l$ to express l_y as a function of l_v such that

$$l_y = \frac{1 - l_v - \Omega^{\frac{1}{\gamma-1}} l_v^{\frac{1-\delta}{1-\gamma}}}{\Upsilon}, \tag{A.7}$$

where $\Upsilon = 1 + \theta(1+\alpha)(1+\xi_c i)$. Substituting (A.7) into (A.2) yields the *general R&D condition*

$$g_A\left\{\frac{1-l_v}{(1+\xi_v i)l_v} - \frac{\Omega^{\frac{1}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}}}{1+\xi_v i} - \frac{\Upsilon[1+\sigma(\frac{1}{1-\alpha}-1)]}{\Gamma\delta\alpha}\right\} = \frac{\sigma\Upsilon(\rho+g_L)}{\Gamma\delta\alpha}. \tag{A.8}$$

In addition, substituting (A.5) into the population-growth condition (30) results in the *population-growth condition*

$$g_L = g_A\left[1 + \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}}\right] \tag{A.9}$$

Consequently, (A.8) and (A.9) represent a system of two equations in two unknowns (l_v and g_A) that can be solved for a balanced-growth equilibrium.

Lemma A.1. *The model has a unique balanced-growth equilibrium. In the equilibrium with a CIA constraint on consumption only, a permanent increase in the nominal interest rate i (a) decreases the fraction of labor allocated to vertical R&D l_v and increases the long-run product-quality growth rate g_A if $\gamma > \delta$, and (b) decreases l_v and g_A if $\gamma < \delta$.*

***Proof of Lemma* A.1.** Imposing $\xi_v = \xi_h = 0$ to reduce (A.5), (A.8) and (A.9) to

$$g_N = \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}} g_A, \tag{A.10}$$

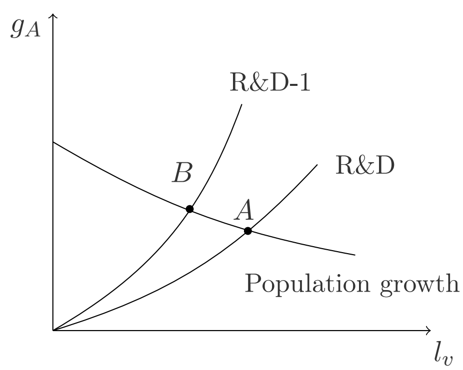
$$g_A\left\{1 - l_v - \Psi^{\frac{1}{\gamma-1}} l_v^{\frac{1-\delta}{1-\gamma}} - \frac{[1+\theta(1+\alpha)(1+\xi_c i)](\Gamma-\sigma)}{\Gamma\delta\alpha} l_v\right\} = \frac{\sigma(\rho+g_L)[1+\theta(1+\alpha)(1+\xi_c i)]}{\Gamma\delta\alpha} l_v \tag{A.11}$$

and

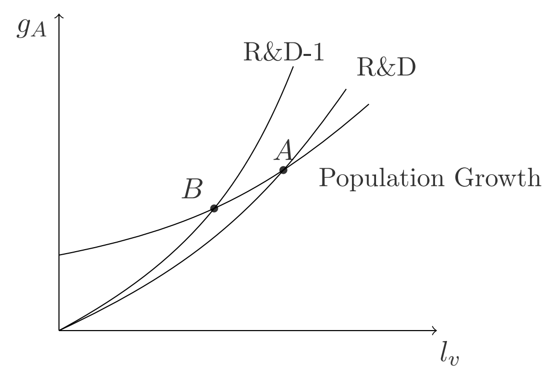
$$g_L = \left[1 + \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}}\right] g_A. \tag{A.12}$$

The last two equations are graphed in Fig.1a assuming that $\gamma > \delta$. The R&D condition curve (A.11) is unambiguously upward sloping and goes through the origin, whereas the *population-growth condition* curve (A.12) is unambiguously downward sloping and has a strictly positive vertical intercept. As illustrated in Fig.1a, there is a unique intersection of these two curves at point A, which pins down the balanced-growth equilibrium values of l_v and g_A. With these values determined, (A.10) pins down g_N, (27) pins down ι, and (28) pins down l_h. Thus, the model has a unique balanced-growth equilibrium when $\gamma > \delta$.

The effect of permanently increasing the nominal interest rate i is illustrated in Fig.1a by the movement from point A to B. An increase in i unambiguously causes the *R&D condition* curve (A.11) to shift up, whereas it has no effect on the *population-growth condition* curve (A.12). Thus, a higher nominal interest rate decreases l_v but increases g_A if $\gamma > \delta$.



(a) $\gamma > \delta$



(b) $\gamma < \delta$

Fig. 1. The effect of a higher nominal interest rate with CIA constraint on consumption.

Equations (A.11) and (A.12) are graphed in Fig.1b assuming $\gamma < \delta$. For $\gamma < \delta$, the slope of the *population-growth condition* curve turns to be positive because a higher l_v is correlated with a higher g_A, whereas the positiveness of the slope of the *general R&D condition* curve remains unchanged. Again, there is a unique intersection of these two curves at point A,[1] which pins down

[1]To show the uniqueness of solution (equilibrium) for equations (A.11) and (A.12), we follow Segerstrom (2000) in rewriting the *general R&D condition* and *population growth condition* as functions of g_N and l_h such that

$$g_N \left\{1 - l_h - \Psi^{\frac{1}{1-\delta}} l_h^{\frac{1-\gamma}{1-\delta}} - \frac{[1+\theta(1+\alpha)(1+\xi_c i)](\Gamma-\sigma)}{\Gamma\delta\alpha}\Psi^{\frac{1}{1-\delta}} l_h^{\frac{1-\gamma}{1-\delta}}\right\} = \frac{(\rho+g_L)[1+\theta(1+\alpha)(1+\xi_c i)]}{\alpha\gamma} l_h,$$

and

$$g_L = \left[1 + \sigma\left(\frac{\lambda_v}{\lambda_h}\right)\Psi^{\frac{\delta}{1-\delta}} l_h^{\frac{\delta-\gamma}{1-\delta}}\right] g_N,$$

respectively, where we have applied (A.4) to express l_v as a function of l_h such that $l_v = \Psi^{\frac{1}{1-\delta}} l_h^{\frac{1-\gamma}{1-\delta}}$ and (A.5). It is straightforward to see that from the first equaiton g_N is unambiguously increasing in l_h and goes through origin, implying a positive slope in (l_h, g_N) space of the *general R&D condition*; g_N in the second equation is unambiguously decreasing in l_h given $\gamma < \delta$ and has a positive vertical intercept, implying a negative slope in (l_h, g_N) space of the

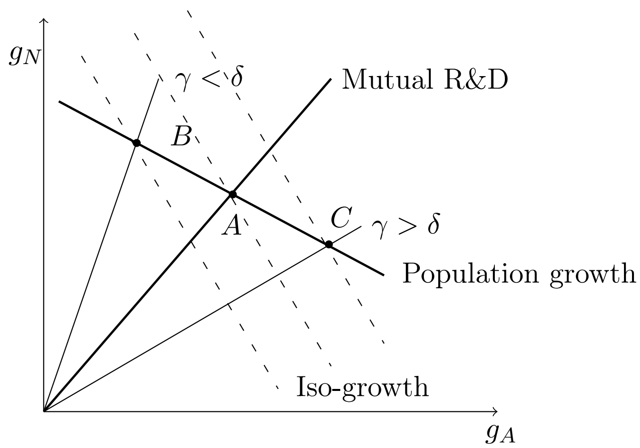


Fig. 2. The growth effect of a higher i with CIA constraint on consumption.

the balanced-growth equilibrium values of l_v and g_A in addition to other variables. The model also has a unique balanced-growth equilibrium if in this case.

The effect of permanently increasing i is illustrated in Fig.1b by moving the equilibrium from point A to B. An increase in i unambiguously shifts the *general R&D condition* curve (A.11) upward, whereas it has no effect on the *population-growth condition* curve (A.12). Therefore, an increase in i decreases l_v and g_A if $\gamma < \delta$. □

Proof of Proposition 1. Based on the above results, we now proceed to the analysis of the overall effects of monetary policies on g_A and g_N. In the (g_A, g_N) space, the slope of each *iso-growth line*(i.e.,$1/(1-\alpha)$) exceeds the slope of the *population-growth condition* (i.e.,1) (in absolute value). The effects of a higher nominal interest rate are illustrated in Fig.2 accordingly. The *mutual R&D condition* given by (A.10) is an upward-sloping line that goes through the origin in the (g_A, g_N) space, when l_v is fixed at the initial equilibrium value. An increase in i shifts down the *mutual R&D condition* to a new intersection C if $\gamma > \delta$, leading to an increase in g_A according to Lemma A.1. In contrast, an increase in i shifts up the *mutual R&D condition* to another new intersection B if $\gamma < \delta$, leading to an decrease in g_A. Combining (29) with (30), one can express the aggregate economic growth rate exclusively as the vertical innovation growth rate such that $g = g_L + [1/(1-\alpha) - 1]g_A$. It implies that an increase in i, which leads to an decrease in g_A when $\gamma < \delta$, decreases the long-run growth rate g (i.e., the movement from A to B); while an increase in i, which results in an increase in g_A when $\gamma > \delta$, increases the long-run growth rate g (i.e., the movement from A to C). □

Lemma A.2. *The model has a unique balanced-growth equilibrium. In the equilibrium with a CIA constraint on vertical R&D only, a permanent increase in i decreases l_v and g_A for both $\gamma > \delta$ and $\gamma < \delta$.*

---

population growth condition. Consequently, there is a unique intersection of these two curves and a unique solution (equilibrium) of these two equations. Given the unique solution of l_h and g_N, $l_v = \Psi^{\frac{1}{1-\delta}} l_h^{\frac{1-\gamma}{1-\delta}}$ from (A.4), and (A.5) immediately imply a unique l_h and g_N, respectively. Hence, the curves illustrated in Fig.1b must intersect once.

***Proof of Lemma* A.2.** Making use of $\xi_c = \xi_h = 0$ to reduce (A.5), (A.8) and (A.9) to

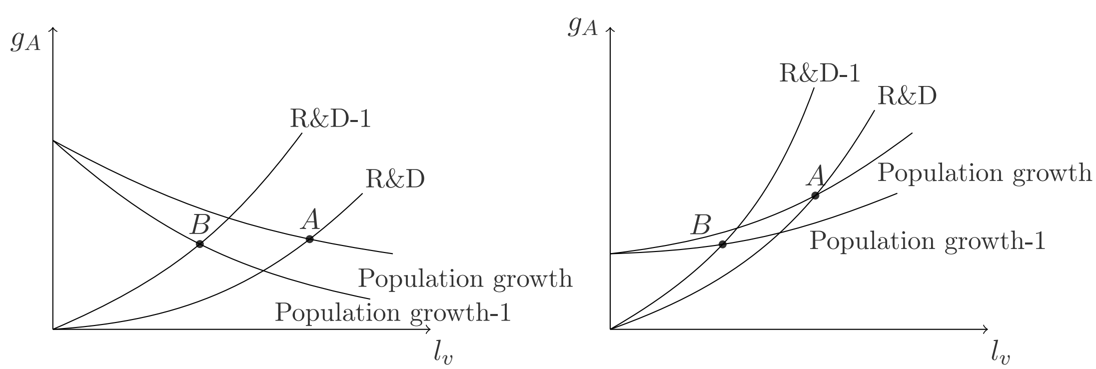
$$g_N = \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}}(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}} g_A, \tag{A.13}$$

$$g_A\left\{\underbrace{\frac{1-l_v}{(1+\xi_v i)l_v} - \Psi^{\frac{1}{\gamma-1}}(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}}_{-} - \frac{(1+\theta+\theta\alpha)(\Gamma-\sigma)}{\Gamma\delta\alpha}\right\} = \frac{\sigma(1+\theta+\theta\alpha)(\rho+g_L)}{\Gamma\delta\alpha}, \tag{A.14}$$

and

$$g_L = \left[1 + \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}}(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}\right] g_A \tag{A.15}$$

Equations (A.14) and (A.15) are graphed in Fig.3a given $\gamma > \delta$. There is a unique intersection of these two curves at point A, which pins down the balanced-growth equilibrium values of all endogenous variables as in the previous case (in which only the CIA constraint on consumption is present). Again, the model has a unique balanced-growth equilibrium when $\gamma > \delta$. The effect of permanently increasing i is illustrated in Fig.3a by the movement from point A to B. A higher i unambiguously causes the *general R&D condition* curve (A.14) (the negative sign means that the value of those terms overall decreases as i increases) to shift upward and the *population-growth condition* curve (A.15) to shift downward. Thus, a higher i surely decreases l_v.



(a) $\gamma > \delta$ (b) $\gamma < \delta$

Fig. 3. The effect of a higher nominal interest rate with CIA constraint on vertical R&D.

As for the effect on g_A, suppose that for some $\gamma > \delta$, an increase in i increases (or has no effect on) g_A. According to (A.15), $(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$ must decrease (or remain unchanged) when i increases, which means that $[(1+\xi_v i)l_v]^{-1} l_v^{\frac{\delta}{\gamma}}$ must increase (or remain unchanged). Given that l_v decreases as i increases, $[(1+\xi_v i)l_v]^{-1}$ must increase in response. Therefore, (A.14) implies that $(1-l_v)/[(1+\xi_v i)l_v] - \Psi^{\frac{1}{\gamma-1}}(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$ must increase and thus g_A must decrease. This yields a contradiction, so g_A must always decrease in a higher i when $\gamma > \delta$.

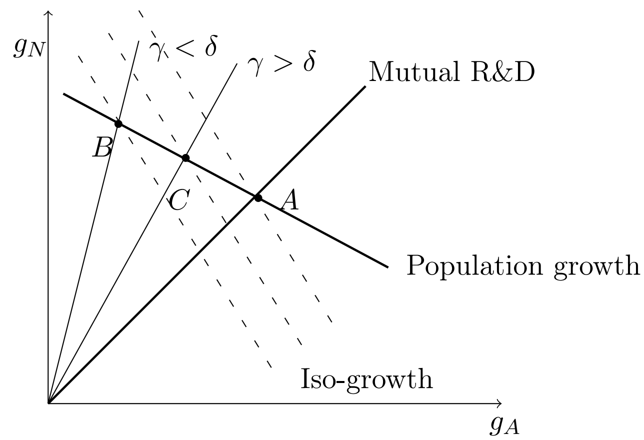


Fig. 4. The growth effect of a higher i with CIA constraint on vertical R&D.

Equations (A.14) and (A.15) for $\gamma < \delta$ are graphed in Fig.3b.[2] There is still a unique intersection of these two curves at point A, so the model has a unique balanced-growth equilibrium when $\gamma < \delta$. The effect of permanently increasing i is illustrated in Fig.3b by the movement from point A to B. An increase in i unambiguously causes the *general R&D condition* curve (A.14) to shift upward, while the *population-growth condition* curve (A.15) to shift downward. Hence, a higher i decreases l_v. A similar proof applies for the change in g_A. □

Proof of Proposition 2. The effects of a higher rate of nominal interest on the aggregate rate of economic growth g are displayed in Fig.4. From Lemma A.2 and (A.15), a decreased g_A due to a rise in i means an increased $(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$. As a result, an increase in i shifts up the *mutual R&D condition* line according to (A.13), implying a lower vertical R&D growth rate for both $\gamma > \delta$ (namely the movement from A to C) and $\gamma < \delta$ (from A to B), with a larger magnitude for the latter case. The difference arises because given a lowered l_v for a rise in i, $\gamma < \delta$ leads $l_v^{\frac{\gamma-\delta}{1-\gamma}}$ to be increasing in i and makes the overall positive effect of a higher i in the term of $(1+\xi_v i)^{\frac{\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$ dominate the one under $\gamma > \delta$ in which $l_v^{\frac{\gamma-\delta}{1-\gamma}}$ is decreasing in i. In other words, the overall effect of a higher nominal interest rate is to increase the product-variety growth rate at the expense of the product-quality growth rate, with a larger sacrifice in vertical innovation growth rate when $\gamma < \delta$. The relation of $g = g_L + [1/(1-\alpha) - 1]g_A$ from (29) and (30) states that a movement on the *population-growth condition* in the northwest direction (g_N increases and g_A decreases) is growth-retarding due to $1 < 1/(1-\alpha)$. Therefore, a larger sacrifice in the product-quality growth rate g_A in the case of $\gamma < \delta$ means a larger decrease in the economic growth rate than that in the case of $\gamma > \delta$. □

[2]The proof of a unique equilibrium is similar to the one shown in Footnote 1.

Lemma A.3. *The model has a unique balanced-growth equilibrium. In the equilibrium with a CIA constraint on horizontal R&D only, a permanent increase in i increases l_v and g_A for both $\gamma > \delta$ and $\gamma > \delta$.*

Proof of Lemma A.3. In an analogous fashion of the proof of Lemma A.2, imposing $\xi_c = \xi_v = 0$ enables us to reduce (A.5), (A.8) and (A.9) to

$$g_N = \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}}(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}} g_A, \tag{A.16}$$

$$g_A\left[\frac{1-l_v}{l_v} - \underbrace{\Psi^{\frac{1}{\gamma-1}}(1+\xi_h i)^{\frac{-1}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}}_{+} - \frac{(1+\theta+\theta\alpha)(\Gamma-\sigma)}{\Gamma\delta\alpha}\right] = \frac{\sigma(1+\theta+\theta\alpha)(\rho+g_L)}{\Gamma\delta\alpha}, \tag{A.17}$$

and

$$g_L = \left[1 + \frac{1}{\sigma}\left(\frac{\lambda_h}{\lambda_v}\right)\Psi^{\frac{\gamma}{\gamma-1}}(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}\right] g_A. \tag{A.18}$$

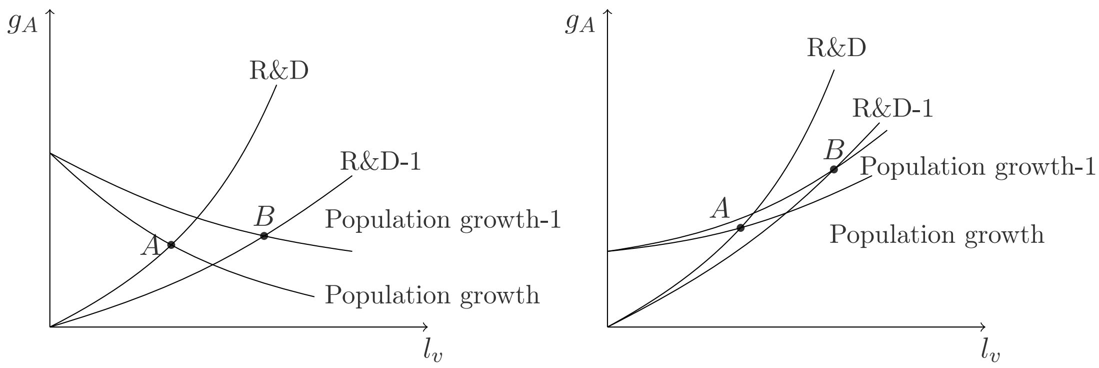
Equations (A.17) and (A.18) are graphed in Fig.5a given $\gamma > \delta$. There is a unique intersection of these two curves at point A, which pins down the balanced-growth equilibrium values of all endogenous variables. The model also has a unique balanced-growth equilibrium when $\gamma > \delta$. The effect of permanently increasing i is illustrated in Fig.5a by the movement from point A to B. An increase in i unambiguously causes the *general R&D condition* curve (A.17) to shift downward and the *population-growth condition* curve (A.18) to shift upward. Hence, a higher i increases l_v.

As for the effect on g_A, suppose that for some $\gamma > \delta$, an increase in i decreases (or does not change) g_A. Then, (A.18) implies that $(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$ increases (or remain constant) when i increases, from which it follows that $[(1+\xi_h i)l_v^{-1}]^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{-\delta}{1-\gamma}}$ increases (or remain constant). Since l_v increases in response to an increase in i, thus $[(1+\xi_h i)l_v^{-1}]^{\frac{-\gamma}{1-\gamma}}$ should increase and $[(1+\xi_h i)l_v^{-1}]$ decrease. From (A.17), $\frac{1-l_v}{l_v} - \Psi^{\frac{1}{\gamma-1}}(1+\xi_h i)^{\frac{-1}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}} = \frac{1}{1+\xi_h i}\left\{\frac{(1+\xi_h i)(1-l_v)}{l_v} - \Psi^{\frac{1}{\gamma-1}}(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}\right\}$ must decrease and g_A must increase. This yields a contradiction. Therefore, g_A must always increase in response to an increase i when $\gamma > \delta$.

Equations (A.17) and (A.18) for $\gamma < \delta$ are graphed in Fig.5b. There is also a unique intersection of these two curves at point A, and the model has a unique balanced-growth equilibrium when $\gamma < \delta$.[3] The effect of a permanent increase in i is illustrated in Fig.5b by the movement from point A to B. An increase in i unambiguously causes the *general R&D condition* curve (A.17) to shift downward and the *population-growth condition* curve (A.18) upward. Thus, a higher i increases l_v. A similar proof applies for the change in g_A. □

Proof of proposition 3. The effects of a higher rate of nominal interest on the aggregate rate of economic growth g are displayed in Fig.6. From Lemma A.3 and (A.18), an increased g_A means a decreased $(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$. As a result, an increase in i shifts down the *mutual R&D condition*

[3] Similarly, see Footnote 1 for the proof of the unique equilibrium in the case of $\gamma < \delta$.



(a) $\gamma > \delta$ (b) $\gamma < \delta$

Fig. 5. The effect of a higher nominal interest rate when CIA constraint on horizontal R&D.

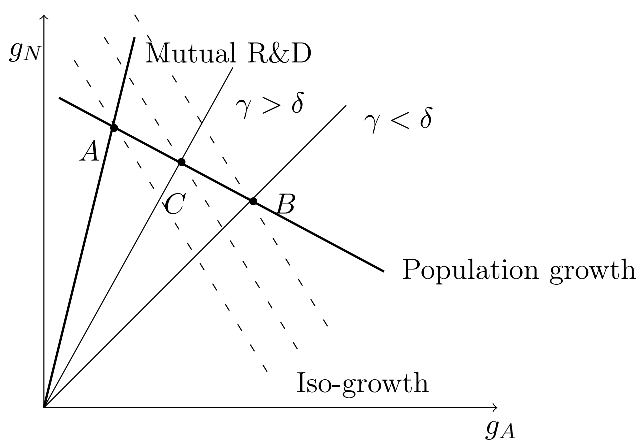


Fig. 6. The growth effect of a higher i with CIA constraint on horizontal R&D.

line according to (A.16), and then increases the vertical R&D growth rate for both $\gamma > \delta$ (the movement from A to C) and $\gamma < \delta$ (from A to B), with a larger magnitude for the latter case again. The difference occurs because given an increased l_v for a higher i, $\gamma < \delta$ leads $l_v^{\frac{\gamma-\delta}{1-\gamma}}$ to be decreasing in i and makes the overall decreasing effect in the term of $(1+\xi_h i)^{\frac{-\gamma}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}$ dominate the one under $\gamma > \delta$ in which $l_v^{\frac{\gamma-\delta}{1-\gamma}}$ is increasing in i. In other words, the overall effect of a higher i is to increase the product-quality growth rate at the cost of the product-variety growth rate, with a larger sacrifice in g_N when $\gamma < \delta$. Again, $g = g_L + [1/(1-\alpha) - 1]g_A$ implies that a movement on the *population-growth condition* in the southeast direction (g_A increases and g_N decreases) is growth-promoting due to $1 < 1/(1-\alpha)$. Therefore, a larger sacrifice in the product-variety growth rate means a larger increase in the aggregate economic growth rate when $\gamma < \delta$. □

A.2 Proof of Proposition 4

To prove Proposition 4, we move one step forward to solve l_v and then the economic growth rate. Given (A.7), (A.2) is used to set up another relation between l_y and l_v to solve for l_v. To do this, ι in (A.2) needs to be eliminated. Rewriting the economic growth rate solely as the vertical innovation growth rate by combining (29) and (30) yields

$$g = g_L + \left(\frac{1}{1-\alpha} - 1\right) g_A.$$

Substituting $g_A = \sigma\lambda_v l_v^\delta \iota$ and $g_N = \lambda_h l_h^\gamma \iota$ into the above equation yields

$$g_L = \iota\left[\sigma\lambda_v l_v^\delta + \lambda_h \Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma(\delta-1)}{\gamma-1}}\right] \tag{A.19}$$

By substituting (A.19) and (A.2), we can reduce ι and express l_y as a function of l_v such that

$$\begin{aligned} l_y &= \frac{(1+\xi_v i)[\rho + g_L + \left(\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}\right) g_A]}{\delta\alpha\Gamma\lambda_v\iota} l_v^{1-\delta} \\ &= \frac{(1+\xi_v i)(\rho+g_L)}{\delta\alpha\Gamma\lambda_v g_L}\left[\sigma\lambda_v l_v^\delta + \lambda_h\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{(\delta-1)\gamma}{\gamma-1}}\right] l_v^{1-\delta} + \frac{(1+\xi_v i)\left(\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}\right)\sigma\lambda_v l_v^\delta \iota}{\delta\alpha\Gamma\lambda_v\iota} l_v^{1-\delta} \\ &= \frac{(1+\xi_v i) l_v}{\delta\alpha\Gamma g_L}\left[\sigma(\rho+g_L) + \sigma g_L\left(\frac{1}{1-\alpha} - 1 + \frac{1}{\sigma}\right)\right] + \frac{(1+\xi_v i)(\rho+g_L)\lambda_h}{\delta\alpha\Gamma g_L \lambda_v}\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\delta-1}{\gamma-1}} \\ &= (1+\xi_v i)\left(\Theta l_v + \frac{\lambda_h\Lambda\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\delta-1}{\gamma-1}}}{\lambda_v}\right) \end{aligned} \tag{A.20}$$

where $\Theta = \frac{\rho\sigma + g_L\Gamma}{\delta\alpha\Gamma g_L}$, $\Lambda = \frac{\rho + g_L}{\delta\alpha\Gamma g_L}$. Substituting (A.20) into (A.6), together with (A.7), to rewrite the labor market-clearing condition as

$$l_v[\Upsilon\Theta(1+\xi_v i) + 1] + \Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{1-\delta}{1-\gamma}}\left[\lambda_h\Upsilon\Lambda(1+\xi_v i)/\lambda_v + \Omega^{-1}\right] = 1. \tag{A.21}$$

Hence, (A.21) implicitly solves l_v.

To find the relation between i and g, we need to derive a function of g exclusively on l_v. Combining (29) with (30), and using the expression of ι yield

$$g = g_L + \frac{\sigma g_L\left(\frac{1}{1-\alpha} - 1\right)}{\sigma + \lambda_h\Omega^{\frac{\gamma}{\gamma-1}} l_v^{\frac{\gamma-\delta}{1-\gamma}}/\lambda_v}. \tag{A.22}$$

Differentiating g with respect to i yields

$$
\begin{aligned}
\frac{\partial g}{\partial i} &= \frac{-\sigma g_L\left(\frac{1}{1-\alpha}-1\right)}{\left(\sigma+\lambda_h\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\delta-\delta}{1-\gamma}}/\lambda_v\right)^2}\left(\frac{\lambda_h}{\lambda_v}\right)\left(\frac{\gamma}{\gamma-1}\Omega^{\frac{1}{\gamma-1}}\frac{\partial\Omega}{\partial i}l_v^{\frac{\gamma-\delta}{1-\gamma}}+\Omega^{\frac{\gamma}{\gamma-1}}\frac{\gamma-\delta}{1-\gamma}\frac{\partial l_v}{\partial i}l_v^{\frac{\gamma-\delta}{1-\gamma}-1}\right)\\
&= \frac{\sigma g_L\lambda_h\left(\frac{1}{1-\alpha}-1\right)\Omega^{\frac{1}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}}{\lambda_v(1-\gamma)\left(\sigma+\lambda_h\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}/\lambda_v\right)^2}\left[\gamma\Psi\frac{\xi_h-\xi_v}{(1+\xi_v i)^2}+(\delta-\gamma)\Psi\frac{1+\xi_h i}{1+\xi_v i}\frac{\frac{\partial l_v}{\partial i}}{l_v}\right]\\
&= \underbrace{\frac{g_L\sigma\delta\Gamma\left(\frac{1}{1-\alpha}-1\right)\Omega^{\frac{1}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}}{(1-\gamma)(1+\xi_v i)^2\left(\sigma+\lambda_h\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}/\lambda_v\right)^2}}_{>0}\left[(\xi_h-\xi_v)+(\delta-\gamma)(1+\xi_v i)(1+\xi_h i)\frac{\frac{\partial l_v}{\partial i}}{\gamma l_v}\right]
\end{aligned}
\tag{A.23}
$$

Therefore, the *sign* of $\partial g/\partial i$ depends on the *sign* of $\left[(\xi_h-\xi_v)+(\delta-\gamma)(1+\xi_v i)(1+\xi_h i)\frac{\partial l_v/\partial i}{\gamma l_v}\right]$. Differentiating (A.21) with respect to i to derive $\partial l_v/\partial i$ (note that Ψ,Θ and Λ are unrelated to i) yields

$$
\begin{aligned}
&\underbrace{\left\{[\Upsilon\Theta(1+\xi_v i)+1]+\frac{1-\delta}{1-\gamma}\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}\left[\frac{\lambda_h\Upsilon\Lambda(1+\xi_v i)}{\lambda_v}+\Omega^{-1}\right]\right\}}_{\chi_1>0}\frac{\partial l_v}{\partial i}\\
&=\left\{(\xi_h-\xi_v)\underbrace{\left[\frac{\gamma\lambda_h\Upsilon\Lambda}{\lambda_v(1-\gamma)(1+\xi_h i)}+\frac{1}{\Psi(1-\gamma)(1+\xi_h i)^2}\right]}_{\chi_2>0}-\underbrace{\frac{\lambda_h\Lambda}{\lambda_v}[\theta\xi_c(1+\alpha)(1+\xi_v i)+\Upsilon\xi_v]}_{\chi_3>0}\right\}\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{1-\delta}{1-\gamma}}\\
&-\underbrace{\Theta[\theta\xi_c(1+\alpha)(1+\xi_v i)+\Upsilon\xi_v]}_{\chi_4>0}l_v\\
&\Leftrightarrow\frac{\partial l_v}{\partial i}=\frac{[(\xi_h-\xi_v)\chi_2-\chi_3]\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{1-\delta}{1-\gamma}}-\chi_4 l_v}{\chi_1}
\end{aligned}
\tag{A.24}
$$

To see how $\left[(\xi_h-\xi_v)+(\delta-\gamma)(1+\xi_v i)(1+\xi_h i)\frac{\partial l_v/\partial i}{\gamma l_v}\right]$ changes in response to i is equivalent to see how the following term changes with i,

$$
(1+\xi_v i)(1+\xi_h i)\frac{\partial l_v/\partial i}{\gamma l_v}=\frac{(1+\xi_v i)(1+\xi_h i)}{\chi_1}\left\{\frac{[(\xi_h-\xi_v)\chi_2-\chi_3]\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}-\chi_4}{\gamma}\right\}. \tag{A.25}
$$

We now show that as $i\to\infty$, (A.25) goes to negative infinity because $\lim_{i\to\infty}(1+\xi_v i)(1+\xi_h i)/\chi_1$ is finite and $\lim_{i\to\infty}\left\{[(\xi_h-\xi_v)\chi_2-\chi_3]\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}-\chi_4\right\}/\gamma=-\infty$.

Firstly, we show that

$$
\begin{aligned}
&\lim_{i\to\infty}\frac{(1+\xi_v i)(1+\xi_h i)}{\chi_1}\\
=&\lim_{i\to\infty}\frac{(1+\xi_v i)(1+\xi_h i)}{\{\Theta[1+\theta(1+\alpha)(1+\xi_c i)](1+\xi_v i)+1\}+\frac{1-\delta}{1-\gamma}\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}\left[\frac{\lambda_h}{\lambda_v}\Lambda[1+\theta(1+\alpha)(1+\xi_c i)](1+\xi_v i)+\Omega^{-1}\right]}\\
=&\lim_{i\to\infty}\frac{1}{\underbrace{\frac{\Theta[1+\theta(1+\alpha)(1+\xi_c i)]}{1+\xi_h i}}_{\kappa_1}+\underbrace{\frac{1}{(1+\xi_v i)(1+\xi_h i)}}_{\kappa_2}+\underbrace{\frac{1-\delta}{1-\gamma}\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}}_{\kappa_3}\left[\underbrace{\frac{\lambda_h\Lambda}{\lambda_v}\frac{1+\theta(1+\alpha)(1+\xi_c i)}{1+\xi_h i}}_{\kappa_4}+\underbrace{\frac{1}{(1+\xi_h i)^2\Psi}}_{\kappa_5}\right]}
\end{aligned}
\tag{A.26}
$$

is finite because as $i\to\infty$, κ_2 and κ_5 monotonically decrease to zero; κ_1 and κ_4 monotonically approach to constant terms of $\theta(1+\alpha)\xi_c/\xi_h$ and $\lambda_h\Lambda\theta(1+\alpha)\xi_c/\xi_h$, respectively, according to L'Hospital's rule; and κ_3 also approaches to a constant.

Secondly, since χ_2 is a monotonically decreasing function of i, and χ_3 and χ_4 are monotonically increasing functions of i, $\lim_{i\to\infty}\left\{[(\xi_h-\xi_v)\chi_2-\chi_3]\,\Omega^{\frac{\gamma}{\gamma-1}}l_v^{\frac{\gamma-\delta}{1-\gamma}}-\chi_4\right\}/\gamma=-\infty$. Therefore, $(1+\xi_v i)(1+\xi_h i)\frac{\partial l_v}{\partial i}$ in (A.25) is monotonically decreasing to a negative infinity and $\lim_{i\to\infty}\partial g/\partial i$ is negative (positive) if $\gamma<(>)\delta$. **(i):** As for $\gamma>\delta$, together with $\xi_h>\xi_v$, $\partial g/\partial i$ is always positive for any $i\geq 0$. **(ii):** As for $\gamma<\delta$, to see whether there exist some i leading to $\partial g/\partial i>0$, one can substitute (A.24) into $\left[(\xi_h-\xi_v)+(\delta-\gamma)(1+\xi_v i)(1+\xi_h i)\frac{\partial l_v/\partial i}{\gamma l_v}\right]$ to show that

$$
\begin{aligned}
&\left(\frac{\partial g}{\partial i}\right)_{i=0}>0\\
\Leftrightarrow&(\xi_h-\xi_v)+(\delta-\gamma)\left\{\frac{\Psi^{\frac{\gamma}{\gamma-1}}l^{\frac{\gamma-\delta}{1-\gamma}}}{\gamma\chi_1}[(\xi_h-\xi_v)\chi_2-\chi_3]-\frac{\chi_4}{\gamma\chi_1}\right\}_{i=0}>0\\
\Leftrightarrow&(\xi_h-\xi_v)>\left\{\frac{(\delta-\gamma)\left(\chi_4+\chi_3\Psi^{\frac{\gamma}{\gamma-1}}l^{\frac{\gamma-\delta}{1-\gamma}}\right)}{\gamma\chi_1+(\delta-\gamma)\chi_2\Psi^{\frac{\gamma}{\gamma-1}}l^{\frac{\gamma-\delta}{1-\gamma}}}\right\}_{i=0}>0,
\end{aligned}
\tag{A.27}
$$

where l_v is determined in (A.21) evaluated at $i=0$. Accordingly, a sufficiently large $(\xi_h-\xi_v)$ is a sufficient and necessary condition for the existence of a local maximum of $g(i)$ for $i\geq 0$. In other words, g is increasing in i for $i<i^*$ and decreasing for $i>i^*$, where i^* can be solved from

$$
(\xi_h-\xi_v)=\frac{(\delta-\gamma)\left(\chi_4+\chi_3\Omega^{\frac{\gamma}{\gamma-1}}l^{\frac{\gamma-\delta}{1-\gamma}}\right)}{\gamma\chi_1+(\delta-\gamma)\chi_2\Omega^{\frac{\gamma}{\gamma-1}}l^{\frac{\gamma-\delta}{1-\gamma}}}.
\tag{A.28}
$$

Online Appendix B : Calibration strategy

In this section, we illustrate the strategy of calibrating the model. Given all predetermined parameters and values, the remaining parameters $\{\lambda_v, \lambda_h, \xi_v, \xi_h, \sigma, \theta\}$ must be assigned. In obtaining these values,[4] we match: (i) the economic growth rate; (ii) the Poisson arrival rate of vertical innovations; (iii) the R&D intensity; (iv) the standard time of employment $l = 1/3$; (v) the population growth rate. The procedures are illustrated as follow.

We first calibrate σ. The equation of economic growth rate is

$$g = g_L + \left(\frac{1}{1-\alpha} - 1\right) g_A. \tag{B.1}$$

Upon selecting the economic growth rate, the population growth rate and α, we then have

$$g_A = \frac{g - g_L}{\frac{1}{1-\alpha} - 1}. \tag{B.2}$$

Once having determined g_A, we use the Poisson arrival rate of vertical innovations to pin down σ such that

$$\sigma = g_A/\phi. \tag{B.3}$$

We next calibrate $\{\xi_v, \xi_h\}$. According to (A.21), l_v is an implicit function of these parameters, so we need to build up three equations and use corresponding empirical moments for calibration. First, we use the R&D intensity indicator. The total R&D expenditure is

$$\text{R\&D expenditure} = w_t L_{vt}(1 + \xi_v i) + w_t L_{ht}(1 + \xi_h i). \tag{B.4}$$

The aggregate GDP is

$$\begin{aligned} GDP &= C(\text{consumption expenditure}) + I(\text{R\&D expenditure}) \\ &= c_t L_t (1 + \xi_c i) + w_t L_{vt}(1 + \xi_v i) + w_t L_{ht}(1 + \xi_h i) \\ &= (1+\alpha)(1 + \xi_c i) w_t L_{yt} + w_t L_{vt}(1 + \xi_v i) + w_t L_{ht}(1 + \xi_h i). \end{aligned} \tag{B.5}$$

Using (B.4) and (B.5) together results in the expression of R&D intensity given by

$$2.6\% = \frac{l_v(1 + \xi_v i) + l_h(1 + \xi_h i)}{(1+\alpha)(1 + \xi_c i) l_y + l_v(1 + \xi_v i) + l_h(1 + \xi_h i)}. \tag{B.6}$$

Rewrite this equation as

$$l_y = \Psi_1 \left[l_v(1 + \xi_v i) + l_h(1 + \xi_h i) \right], \tag{B.7}$$

where

$$\Psi_1 = \frac{1 - 2.6\%}{2.6\%(1+\alpha)(1 + \xi_c i)}$$

[4] As explained in the article, λ_v is normalized to one and λ_h is chosen as a free parameter for ensuring the reasonable values of the remaining four parameters.

is known for α, ξ_c and the benchmark nominal interest rate i have been chosen. Another equation making use of the empirical moment of the standard time of employment is given by

$$l = 1/3 = l_y + l_v + l_h. \tag{B.8}$$

Equations (B.7) and (B.8) show that

$$l_h = \frac{1/3 - [1 + \Psi_1(1 + \xi_v i)]l_v}{1 + \Psi_1(1 + \xi_h i)}. \tag{B.9}$$

Together with

$$l_h = \left(\frac{1 + \xi_v i}{1 + \xi_h i}\right)^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{\delta\Gamma}\right)^{\frac{1}{1-\gamma}} \left(\frac{\lambda_h}{\lambda_v}\right)^{\frac{1}{1-\gamma}} l_v^{\frac{1-\delta}{1-\gamma}}, \tag{B.10}$$

the first equation used for pinning down the unknowns $\{\xi_v, \xi_h, l_v\}$ is given by

$$\frac{1/3 - [1 + \Psi_1(1 + \xi_v i)]l_v}{1 + \Psi_1(1 + \xi_h i)} = \left(\frac{1 + \xi_v i}{1 + \xi_h i}\right)^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{\delta\Gamma}\right)^{\frac{1}{1-\gamma}} \left(\frac{\lambda_h}{\lambda_v}\right)^{\frac{1}{1-\gamma}} l_v^{\frac{1-\delta}{1-\gamma}}. \tag{B.11}$$

The second equation for solving $\{\xi_v, \xi_h, l_v\}$ is

$$\frac{g_N}{g_A} = \frac{1}{\sigma} \left(\frac{1 + \xi_v i}{1 + \xi_h i}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{\delta\Gamma}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\lambda_h}{\lambda_v}\right)^{\frac{1}{1-\gamma}} l_v^{\frac{\gamma-\delta}{1-\gamma}}, \tag{B.12}$$

where $\Gamma = 1 + \frac{\sigma}{1-\alpha}$ is now known once σ and α are determined. The last equation is

$$\begin{aligned} l_y &= (1 + \xi_v i) \left[\left(\frac{\rho\sigma + g_L\Gamma}{\delta\alpha\Gamma g_L}\right) l_v + \left(\frac{\rho + g_L}{\delta\alpha\Gamma g_L}\right) \left(\frac{1 + \xi_v i}{1 + \xi_h i}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{\delta\Gamma}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\lambda_h}{\lambda_v}\right)^{\frac{1}{1-\gamma}} l_v^{\frac{1-\delta}{1-\gamma}} \right] \\ &= \Psi_1 \left[l_v(1 + \xi_v i) + (1 + \xi_h i) \frac{1/3 - [1 + \Psi_1(1 + \xi_v i)]l_v}{1 + \Psi_1(1 + \xi_h i)} \right] \\ \Leftrightarrow & \left[\left(\frac{\rho\sigma + g_L\Gamma}{\delta\alpha\Gamma g_L}\right) l_v + \left(\frac{\rho + g_L}{\delta\alpha\Gamma g_L}\right) \left(\frac{1 + \xi_v i}{1 + \xi_h i}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{\delta\Gamma}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\lambda_h}{\lambda_v}\right)^{\frac{1}{1-\gamma}} l_v^{\frac{1-\delta}{1-\gamma}} \right] \\ &= \Psi_1 \left[1 - \frac{1 + \Psi_1(1 + \xi_v i)}{1 + \Psi_1(1 + \xi_h i)} \frac{1 + \xi_h i}{1 + \xi_v i} \right] l_v + \Psi_1 \frac{1 + \xi_h i}{1 + \xi_v i} \frac{1/3}{1 + \Psi_1(1 + \xi_h i)}. \end{aligned} \tag{B.13}$$

Eventually, we have three equations (B.11), (B.12) and (B.13), and three unknowns $\{\xi_v, \xi_h, l_v\}$.

Having found these calibrated values, we thereafter obtain l_y and then θ by solving

$$w_t(1 - l) = \theta(1 + \alpha)(1 + \xi_c i)c_t \Leftrightarrow 1 - l = 2/3 = \theta(1 + \alpha)(1 + \xi_c i)l_y. \tag{B.14}$$

Online Appendix C : Stability analysis

C.1 Characterization of the dynamic system

Before establishing the dynamic system, we claim that the relative productivity parameter $z_{it} \equiv A_{it}/A_t$ in equation (22) in our paper follows the distribution of $Pr\{z_{it} \leq z\} \equiv F(z) = z^{1/\sigma}$ at any time. As shown in Howitt (1999) and Segerstrom (2000), the leading-edge productivity parameter A_t is sufficiently large at the initial steady-state so that the relative productivity parameter converges to the invariant distribution, which implies $\Pi_{ht} = \Pi_{vt}/\Gamma$. Thereafter, to characterize the dynamic system, we first redefine ι_t, which represents the aggregate quality-adjusted labor force, as

$$z_1 \equiv \frac{L_t}{A_t N_t}.$$

We next define the aggregate technology level $T_t = A_t^{1/(1-\alpha)} N_t$ and then have

$$z_2 \equiv \frac{a_t}{T_t}; \quad z_3 \equiv \frac{c_t}{T_t} = \frac{1-\alpha^2}{\Gamma} l_{yt},$$

where we have used $c_t = \frac{(1-\alpha^2) l_{yt} A_t^{\frac{1}{1-\alpha}} N_t}{\Gamma}$ from (25). Denote the economic growth rate $g_t \equiv \dot{T}_t/T_t$. Thus, taking log of z_3 and differentiating it with respect to time yields the motion of z_3 given by

$$\frac{\dot{z}_3}{z_3} = r_t - g_L - \rho - g_t = \frac{\dot{l}_{yt}}{l_{yt}}, \tag{C.1}$$

where the Euler equation is applied. Moreover, recall from the households' budget constraint

$$\dot{a}_t + \dot{m}_t = (r_t - g_L) a_t + w_t l_t + i b_t + \zeta_t - (\pi_t + g_L) m_t - c_t + d_t. \tag{C.2}$$

Using the asset market-clearing condition, the bond market-clearing condition, the government budget constraint, the CIA constraint, the households' optimal decision on leisure, and the expression of d_t:

$$a_t L_t = N_t \Pi_{ht}; \quad b_t L_t = \xi_v w_t L_{vt} + \xi_h w_t L_{ht}; \quad \dot{m}_t + (\pi_t + g_L) m_t = \zeta_t; \quad \xi_c c_t + b_t = m_t,$$

$$w_t (1 - l_t) = \theta c_t (1 + \xi_c i); \quad d_t L_t = (1-\delta) \phi_t \Pi_{vt} N_t + (1-\gamma) \dot{N}_t \Pi_{ht},$$

(C.2) is reduced to

$$\dot{a}_t = (r_t - g_L) a_t + w_t [1 + i(\xi_v l_{vt} + \xi_h l_{ht})] - c_t [1 + \theta(1 + \xi_c i)] + d_t. \tag{C.3}$$

With (C.3), taking log of z_2 and differentiating it with respect to time yields the motion of z_2:

$$\begin{aligned}\frac{\dot{z}_2}{z_2} &= \frac{\dot{a}_t}{a_t} - g_t \\ &= r_t - g_L - g_t + \frac{w_t[1+i(\xi_v l_{vt} + \xi_h l_{ht})]}{a_t} - \frac{c_t[1+\theta(1+\xi_c i)]}{a_t} + \frac{d_t}{a_t} \\ &= \rho + \frac{\dot{z}_3}{z_3} + \frac{(1-\alpha)[1+i(\xi_v l_{vt} + \xi_h l_{ht})]}{\Gamma z_2} - \frac{z_3[1+\theta(1+\xi_c i)]}{z_2} + \frac{(1-\delta)\phi_t \Pi_{vt} N_t + (1-\gamma)\dot{N}_t \Pi_{ht}}{N_t \Pi_{ht}} \\ &= \rho + \frac{\dot{z}_3}{z_3} + \frac{(1-\alpha)[1+i(\xi_v l_{vt} + \xi_h l_{ht})]}{\Gamma z_2} - \frac{z_3[1+\theta(1+\xi_c i)]}{z_2} + [\Gamma(1-\delta)\lambda_v l_{vt}^{\delta} + (1-\gamma)\lambda_h l_{ht}^{\gamma}] z_1,\end{aligned} \tag{C.4}$$

where we have used (C.1) and the relations

$$\frac{w_t}{a_t} = \frac{w_t}{T_t}\frac{T_t}{a_t} = \frac{1-\alpha}{\Gamma z_2}, \quad \frac{c_t}{a_t} = \frac{c_t}{T_t}\frac{T_t}{a_t} = \frac{z_3}{z_2}, \quad \phi = \lambda_v \iota_t l_{vt}^{\delta} = \lambda_v z_1 l_{vt}^{\delta}$$

$$g_{Nt} = \lambda_h \iota_t l_{ht}^{\gamma} = \lambda_h z_1 l_{ht}^{\gamma}, \quad a_t L_t = N_t \Pi_{ht}, \quad \Pi_{ht} = \Gamma^{-1}\Pi_{vt}.$$

Similarly, the motion of z_1 is

$$\frac{\dot{z}_1}{z_1} = g_L - \frac{\dot{A}_t}{A_t} - \frac{\dot{N}_t}{N_t} = g_L - (\sigma\lambda_v l_{vt}^{\delta} + \lambda_h l_{ht}^{\gamma}) z_1, \tag{C.5}$$

where we have used the equation $A_t = \sigma\phi_t = \sigma\lambda_v l_{vt}^{\delta}\iota_t$ in the derivation of the second equality.

The economic system is now preliminarily established by the differential equations (C.1), (C.4) and (C.5). The next step is to replace the endogenous variables l_{vt}, l_{ht} and l_{yt}. Firstly, using the first-order conditions determining the optimal labor allocations in both vertical and horizontal R&D sectors

$$\frac{\lambda_v \delta \Pi_{vt}}{A_t} l_{vt}^{\delta-1} = w_t(1+\xi_v i), \tag{C.6}$$

and

$$\frac{\lambda_h \gamma \Pi_{ht}}{A_t} l_{ht}^{\gamma-1} = w_t(1+\xi_h i), \tag{C.7}$$

we can express l_{ht} as a function of l_{vt} given by

$$l_{ht} = \Omega^{\frac{1}{\gamma-1}} l_{vt}^{\frac{1-\delta}{1-\gamma}}. \tag{C.8}$$

Secondly, using the $a_t L_t = N_t \Pi_{ht}$, $w_t = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}} N_t}{\Gamma} = \frac{(1-\alpha)T_t}{\Gamma}$, (C.6) and (C.7), we have

$$\begin{aligned} a_t &= \frac{N_t \Pi_{ht}}{L_t} = \frac{N_t \Pi_{vt}}{\Gamma L_t} = \frac{N_t}{L_t}\frac{A_t w_t (1+\xi_v i)}{\delta \lambda_v \Gamma} l_{vt}^{1-\delta} = \frac{1+\xi_v i}{\delta\lambda_v \Gamma}\frac{w_t}{z_1} l_{vt}^{1-\delta} \\ \Leftrightarrow l_{vt}^{1-\delta} &= \frac{a_t}{w_t}\frac{\delta\lambda_v\Gamma z_1}{1+\xi_v i} = \frac{T_t z_2}{w_t}\frac{\delta\lambda_v\Gamma z_1}{1+\xi_v i} = \frac{z_2}{\frac{1-\alpha}{\Gamma}}\frac{\delta\lambda_v\Gamma z_1}{1+\xi_v i} \\ \Leftrightarrow l_{vt} &= \Xi_1^{\frac{1}{1-\delta}} (z_1 z_2)^{\frac{1}{1-\delta}}, \end{aligned} \tag{C.9}$$

where

$$\Xi_1 = \frac{\lambda_v \delta \Gamma^2}{(1-\alpha)(1+\xi_v i)}.$$

Use (C.9) to rewrite (C.8) more compactly as

$$l_{ht} = \Xi_2^{\frac{1}{1-\gamma}} (z_1 z_2)^{\frac{1}{1-\gamma}}, \tag{C.10}$$

where

$$\Xi_2 = \frac{\Xi_1}{\Omega} = \frac{\lambda_h \gamma \Gamma}{(1-\alpha)(1+\xi_h i)}.$$

By substituting (C.9) and (C.10) into (C.5), we obtain the first differential equation governing the dynamic system given by

$$\frac{\dot{z}_1}{z_1} = g_L - \left[\sigma \lambda_v \Xi_1^{\frac{\delta}{1-\delta}} (z_1 z_2)^{\frac{\delta}{1-\delta}} + \lambda_h \Xi_2^{\frac{\gamma}{1-\gamma}} (z_1 z_2)^{\frac{\gamma}{1-\gamma}}\right]. \tag{C.11}$$

To derive the second differential equation, we again substitute (C.9) and (C.10) into (C.4) and multiply both sides of it with z_2, which yields

$$\begin{aligned}
&\dot{z}_2 + [1+\theta(1+\xi_c i)]z_3 - \rho z_2 - \frac{\dot{z}_3}{z_3} z_2 \\
=&\frac{1-\alpha}{\Gamma}\left\{1 + i\left[\xi_v \Xi_1^{\frac{1}{1-\delta}} (z_1 z_2)^{\frac{1}{1-\delta}} + \xi_h \Xi_2^{\frac{1}{1-\gamma}} (z_1 z_2)^{\frac{1}{1-\gamma}}\right]\right\} \\
&+ \left[\Gamma(1-\delta)\lambda_v \Xi^{\frac{\delta}{1-\delta}} (z_1 z_2)^{\frac{\delta}{1-\delta}} + \lambda_h (1-\gamma) \Xi_2^{\frac{\gamma}{1-\gamma}} (z_1 z_2)^{\frac{\gamma}{1-\gamma}}\right](z_1 z_2) \\
=&\frac{1-\alpha}{\Gamma} + \left[\frac{(1-\alpha)\xi_v i}{\Gamma} \Xi_1^{\frac{1}{1-\delta}} + \lambda_v \Gamma (1-\delta) \Xi_1^{\frac{\delta}{1-\delta}}\right] (z_1 z_2)^{\frac{1}{1-\delta}} \\
&+ \left[\frac{(1-\alpha)\xi_h i}{\Gamma} \Xi_2^{\frac{1}{1-\gamma}} + \lambda_h (1-\gamma) \Xi_2^{\frac{\gamma}{1-\gamma}}\right] (z_1 z_2)^{\frac{1}{1-\gamma}} \\
=&\frac{1-\alpha}{\Gamma}\left[1 + \left(\frac{1-\delta+\xi_v i}{\delta}\right) \Xi_1^{\frac{1}{1-\delta}} (z_1 z_2)^{\frac{1}{1-\delta}} + \frac{(1-\alpha)(1-\gamma+\xi_h i)}{\gamma \Gamma} \Xi_2^{\frac{1}{1-\gamma}} (z_1 z_2)^{\frac{1}{1-\gamma}}\right].
\end{aligned} \tag{C.12}$$

We now have two differential equations yet three endogenous variables. We then need another equation to complete the description of dynamic system. Substituting $w_t(1-l_t) = c_t(1+\xi_c i)$, the expressions of c_t and w_t, (C.9) and (C.10) into the labor market-clearing condition $l_t = l_{yt} + l_{vt} + l_{ht}$ yields

$$\begin{aligned}
&1 - \theta(1+\alpha)(1+\xi_c i) l_{yt} = l_{yt} + l_{vt} + l_{ht} \\
\Leftrightarrow & z_3 = \Xi_3 \left[1 - \Xi_1^{\frac{1}{1-\delta}} (z_1 z_2)^{\frac{1}{1-\delta}} - \Xi_2^{\frac{1}{1-\gamma}} (z_1 z_2)^{\frac{1}{1-\gamma}}\right],
\end{aligned} \tag{C.13}$$

where

$$\Xi_3 = \frac{1-\alpha^2}{\Gamma[1+\theta(1+\alpha)(1+\xi_c i)]}.$$

Differentiating z_3 with respect to time yields

$$\begin{aligned}
\dot{z}_3 &= -\Xi_3\left[\Xi_1^{\frac{1}{1-\delta}}\left(\frac{1}{1-\delta}\right)(z_1z_2)^{\frac{1}{1-\delta}-1}(\dot{z}_1z_2+\dot{z}_2z_1) + \Xi_2^{\frac{1}{1-\gamma}}\left(\frac{1}{1-\gamma}\right)(z_1z_2)^{\frac{1}{1-\gamma}-1}(\dot{z}_1z_2+\dot{z}_2z_1)\right] \\
&= -\Xi_3\left[\frac{\Xi_1^{\frac{1}{1-\delta}}}{1-\delta}(z_1z_2)^{\frac{1}{1-\delta}} + \frac{\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma}(z_1z_2)^{\frac{1}{1-\gamma}}\right]\left(\frac{\dot{z}_1}{z_1}+\frac{\dot{z}_2}{z_2}\right).
\end{aligned} \tag{C.14}$$

Substituting (C.13) and (C.14) into (C.12) to reduce $\dot{z}_3$ and z_3 yields

$$\begin{aligned}
\dot{z}_2 &= \rho z_2 - \Xi_3[1+\theta(1+\xi_c i)]\left[1-\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} - \Xi_2^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}}\right] \\
&\quad + z_2\left(\frac{\dot{z}_1}{z_1}+\frac{\dot{z}_2}{z_2}\right)\frac{\frac{\Xi_1^{\frac{1}{1-\delta}}}{1-\delta}(z_1z_2)^{\frac{1}{1-\delta}} + \frac{\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma}(z_1z_2)^{\frac{1}{1-\gamma}}}{\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} + \Xi_2^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}} - 1} \\
&\quad + \frac{1-\alpha}{\Gamma}\left[1+\left(\frac{1-\delta+\xi_v i}{\delta}\right)\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} + \left(\frac{1-\gamma+\xi_h i}{\gamma}\right)\Xi_1^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}}\right] \\
\Leftrightarrow \dot{z}_2 &\left[\frac{1+\frac{\delta\Xi_1^{\frac{1}{1-\delta}}}{1-\delta}(z_1z_2)^{\frac{1}{1-\delta}} + \frac{\gamma\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma}(z_1z_2)^{\frac{1}{1-\gamma}}}{1-\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} - \Xi_2^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}}}\right] \\
&= \rho z_2 - \Xi_3[1+\theta(1+\xi_c i)]\left[1-\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} - \Xi_2^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}}\right] \\
&\quad + z_2\left[\frac{\frac{\Xi_1^{\frac{1}{1-\delta}}}{1-\delta}(z_1z_2)^{\frac{1}{1-\delta}} + \frac{\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma}(z_1z_2)^{\frac{1}{1-\gamma}}}{\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} + \Xi_2^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}} - 1}\right]\frac{\dot{z}_1}{z_1} \\
&\quad + \frac{1-\alpha}{\Gamma}\left[1+\left(\frac{1-\delta+\xi_v i}{\delta}\right)\Xi_1^{\frac{1}{1-\delta}}(z_1z_2)^{\frac{1}{1-\delta}} + \left(\frac{1-\gamma+\xi_h i}{\gamma}\right)\Xi_1^{\frac{1}{1-\gamma}}(z_1z_2)^{\frac{1}{1-\gamma}}\right]
\end{aligned} \tag{C.15}$$

Finally, the dynamic system is represented by two variables z_1 and z_2 and their differential equations of (C.11) and (C.15).

C.2 Steady-state values of z_1 and z_2

The steady-state values of variables z_1 and z_2 (i.e., z_1^* and z_2^*) are solved respectively by using (C.9) and (C.10). Given the calibrated parameters, $l_v = 0.0102$ and $l_h = 0.00014$. Substituting them into (C.9) and (C.10) eventually yields $z_1 = 0.1811$ and $z_2 = 12.5141$.

C.3 Linearization

Once obtaining the steady-state values of variables z_1 and z_2, we can linearize the above nonlinear dynamic system around z_1^* and z_2^*. Formally, the Taylor series expansion of the nonlinear system around the steady-state is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \mathbf{J} \cdot \begin{bmatrix} z_1 - z_1^* \\ z_2 - z_2^* \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} z_1 - z_1^* \\ z_2 - z_2^* \end{bmatrix}$$

where $\mathbf{J}$ is the corresponding Jacobian matrix with

$$J_{11} = \frac{\partial \dot{z}_1}{\partial z_1}|_{(z_1^*, z_2^*)} = g_L - z_1^* \left[\frac{\sigma \lambda_v (2-\delta)}{1-\delta} \Xi_1^{\frac{\delta}{1-\delta}} (z_1^* z_2^*)^{\frac{\delta}{1-\delta}} + \frac{\lambda_h (2-\gamma)}{1-\gamma} \Xi_2^{\frac{\gamma}{1-\gamma}} (z_1^* z_2^*)^{\frac{\gamma}{1-\gamma}} \right], \tag{C.16}$$

and

$$J_{12} = \frac{\partial \dot{z}_1}{\partial z_2}|_{(z_1^*, z_2^*)} = - \frac{(z_1^*)^2}{z_2^*} \left[\frac{\sigma \lambda_v \delta}{1-\delta} \Xi_1^{\frac{\delta}{1-\delta}} (z_1^* z_2^*)^{\frac{\delta}{1-\delta}} + \frac{\lambda_h \gamma}{1-\gamma} \Xi_2^{\frac{\delta}{1-\delta}} (z_1^* z_2^*)^{\frac{\gamma}{1-\gamma}} \right]. \tag{C.17}$$

To derive J_{21} and J_{22}, we rewrite (C.15) as

$$\begin{aligned}
&\dot{z}_2 \left(\frac{1 + \frac{\delta}{1-\delta} l_{vt} + \frac{\gamma}{1-\gamma} l_{ht}}{1 - l_{vt} - l_{ht}} \right) = \rho z_2 - \Xi_3 [1 + \theta(1 + \xi_c i)](1 - l_{vt} - l_{ht}) \\
&+ z_2 \left(\frac{\frac{\delta}{1-\delta} l_{vt} + \frac{\gamma}{1-\gamma} l_{ht}}{l_{vt} + l_{ht} - 1} \right) \frac{\dot{z}_1}{z_1} + \frac{1-\alpha}{\Gamma} \left[1 + \frac{1 - \delta + \xi_v i}{\delta} l_{vt} + \frac{1 - \gamma + \xi_h i}{\gamma} l_{ht} \right]
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow &\dot{z}_2 \underbrace{\left(1 + \frac{\delta}{1-\delta} l_{vt} + \frac{\gamma}{1-\gamma} l_{ht} \right)}_{\text{term 1}} = \rho z_2 (1 - l_{vt} - l_{ht}) - \Xi_3 [1 + \theta(1 + \xi_c i)](1 - l_{vt} - l_{ht})^2 \\
&- z_2 \left(\frac{l_{vt}}{1-\delta} + \frac{1 - l_{ht}}{1-\gamma} \right) \frac{\dot{z}_1}{z_1} + \frac{1-\alpha}{\Gamma} \left[1 + \frac{1 - \delta + \xi_v i}{\delta} l_{vt} + \frac{1 - \gamma + \xi_h i}{\gamma} l_{ht} \right] (1 - l_{vt} - l_{ht})
\end{aligned} \tag{C.18}$$

$$\begin{aligned}
= &\underbrace{\rho z_2 (1 - l_{vt} - l_{ht})}_{\text{term 2}} - \underbrace{\Xi_3 [1 + \theta(1 + \xi_c i)](1 - l_{vt} - l_{ht})^2}_{\text{term 3}} \\
&- \underbrace{\left(\frac{l_{vt}}{1-\delta} + \frac{1 - l_{ht}}{1-\gamma} \right) \left[g_L z_2 - \left(\frac{\sigma \lambda_v l_{vt}}{\Xi_1} + \frac{\lambda_h l_{ht}}{\Xi_2} \right) \right]}_{\text{term 4}} \\
&+ \underbrace{\frac{1-\alpha}{\Gamma} \left[1 + \frac{1 - \delta + \xi_v i}{\delta} l_{vt} + \frac{1 - \gamma + \xi_h i}{\gamma} l_{ht} \right] (1 - l_{vt} - l_{ht})}_{\text{term 5}}
\end{aligned}$$

Alternatively,

$$\dot{z}_2 = \frac{\text{term 2} - \text{term 3} - \text{term 4} + \text{term 5}}{\text{term 1}}.$$

Therefore, differentiating $\dot{z}_2$ with respect to z_1 and z_2 respectively yields

$$J_{21} = \frac{\partial \dot{z}_2}{\partial z_1}|_{(z_1^*, z_2^*)}$$
$$= \frac{\left(\frac{\partial \text{term 2}}{\partial z_1} - \frac{\partial \text{term 3}}{\partial z_1} - \frac{\partial \text{term 4}}{\partial z_1} + \frac{\partial \text{term 5}}{\partial z_1}\right) \cdot \text{term 1} - \frac{\partial \text{term 1}}{\partial z_1} \cdot (\text{term 2} - \text{term 3} - \text{term 4} + \text{term 5})}{(\text{term 1})^2}$$

$$J_{22} = \frac{\partial \dot{z}_2}{\partial z_2}|_{(z_1^*, z_2^*)}$$
$$= \frac{\left(\frac{\partial \text{term 2}}{\partial z_2} - \frac{\partial \text{term 3}}{\partial z_2} - \frac{\partial \text{term 4}}{\partial z_2} + \frac{\partial \text{term 5}}{\partial z_2}\right) \cdot \text{term 1} - \frac{\partial \text{term 1}}{\partial z_2} \cdot (\text{term 2} - \text{term 3} - \text{term 4} + \text{term 5})}{(\text{term 1})^2}.$$

The corresponding partial derivatives, evaluated at the steady-state (i.e., z_1^* and z_2^*), are given by

$$\frac{\partial \text{term 1}}{\partial z_1} = \frac{\delta}{1-\delta}\frac{\partial l_{vt}}{\partial z_1} + \frac{\gamma}{1-\gamma}\frac{\partial l_{ht}}{\partial z_1}, \quad \frac{\partial \text{term 2}}{\partial z_1} = -\rho z_2 \left(\frac{\partial l_{vt}}{\partial z_1} + \frac{\partial l_{ht}}{\partial z_1}\right),$$
$$\frac{\partial \text{term 3}}{\partial z_1} = -2\Xi_3[1+\theta(1+\xi_c i)](1-l_{vt}-l_{ht})\left(\frac{\partial l_{vt}}{\partial z_1} + \frac{\partial l_{ht}}{\partial z_1}\right),$$

$$\frac{\partial \text{term 4}}{\partial z_1} = \left(\frac{1}{1-\delta}\frac{\partial l_{vt}}{\partial z_1} + \frac{1}{1-\gamma}\frac{\partial l_{ht}}{\partial z_1}\right)\left(g_L z_2 - \frac{\lambda_v \sigma l_{vt}}{\Xi_1}\frac{\partial l_{vt}}{\partial z_1} - \frac{\lambda_h}{\Xi_2}\frac{\partial l_{ht}}{\partial z_1}\right)$$
$$+ \left(\frac{l_{vt}}{1-\delta} + \frac{l_{ht}}{1-\gamma}\right)\left(g_L z_2 - \frac{\lambda_v \sigma l_{vt}}{\Xi_1}\frac{\partial l_{vt}}{\partial z_1} - \frac{\lambda_h}{\Xi_2}\frac{\partial l_{ht}}{\partial z_1}\right)$$

$$\frac{\partial \text{term 5}}{\partial z_1} = \frac{1-\alpha}{\Gamma}\left[\left(\frac{1-\delta+\xi_v i}{\delta}\right)\frac{\partial l_{vt}}{\partial z_1} + \left(\frac{1-\gamma+\xi_h i}{\gamma}\frac{\partial l_{ht}}{\partial z_1}\right)\right](1-l_{vt}-l_{ht})$$
$$- \frac{1-\alpha}{\Gamma}\left[1 + \left(\frac{1-\delta+\xi_v i}{\delta}\right)l_{vt} + \left(\frac{1-\gamma+\xi_h i}{\gamma}l_{ht}\right)\right]\left(\frac{\partial l_{vt}}{\partial z_1} + \frac{\partial l_{ht}}{\partial z_1}\right),$$

and

$$\frac{\partial \text{term 1}}{\partial z_2} = \frac{\delta}{1-\delta}\frac{\partial l_{vt}}{\partial z_2} + \frac{\gamma}{1-\gamma}\frac{\partial l_{ht}}{\partial z_2}, \quad \frac{\partial \text{term 2}}{\partial z_2} = -\rho(1-l_{vt}-l_{ht}) - \rho z_2 \left(\frac{\partial l_{vt}}{\partial z_2} + \frac{\partial l_{ht}}{\partial z_2}\right),$$

$$\frac{\partial \text{term 3}}{\partial z_1} = -2\Xi_3[1+\theta(1+\xi_c i)](1-l_{vt}-l_{ht})\left(\frac{\partial l_{vt}}{\partial z_1} + \frac{\partial l_{ht}}{\partial z_1}\right),$$

$$\frac{\partial \text{term 4}}{\partial z_2} = \left(\frac{1}{1-\delta}\frac{\partial l_{vt}}{\partial z_2} + \frac{1}{1-\gamma}\frac{\partial l_{ht}}{\partial z_2}\right)\left(g_L z_2 - \frac{\lambda_v \sigma l_{vt}}{\Xi_1}\frac{\partial l_{vt}}{\partial z_2} - \frac{\lambda_h}{\Xi_2}\frac{\partial l_{ht}}{\partial z_2}\right)$$
$$+ \left(\frac{l_{vt}}{1-\delta} + \frac{l_{ht}}{1-\gamma}\right)\left(g_L z_2 - \frac{\lambda_v \sigma l_{vt}}{\Xi_1}\frac{\partial l_{vt}}{\partial z_2} - \frac{\lambda_h}{\Xi_2}\frac{\partial l_{ht}}{\partial z_2}\right)$$

$$\frac{\partial \text{term 5}}{\partial z_2} = \frac{1-\alpha}{\Gamma}\left[\left(\frac{1-\delta+\xi_v i}{\delta}\right)\frac{\partial l_{vt}}{\partial z_2} + \left(\frac{1-\gamma+\xi_h i}{\gamma}\frac{\partial l_{ht}}{\partial z_2}\right)\right](1-l_{vt}-l_{ht})$$
$$- \frac{1-\alpha}{\Gamma}\left[1 + \left(\frac{1-\delta+\xi_v i}{\delta}\right)l_{vt} + \left(\frac{1-\gamma+\xi_h i}{\gamma}l_{ht}\right)\right]\left(\frac{\partial l_{vt}}{\partial z_2} + \frac{\partial l_{ht}}{\partial z_2}\right),$$

where

$$l_{vt} = \Xi_1^{\frac{1}{1-\delta}} (z_1 z_2)^{\frac{1}{1-\delta}}; \quad \frac{\partial l_{vt}}{\partial z_1} = \frac{\Xi_1^{\frac{1}{1-\delta}}}{1-\delta} z_1^{\frac{\delta}{1-\delta}} z_2^{\frac{1}{1-\delta}}; \quad \frac{\partial l_{vt}}{\partial z_2} = \frac{\Xi_1^{\frac{1}{1-\delta}}}{1-\delta} z_2^{\frac{\delta}{1-\delta}} z_1^{\frac{1}{1-\delta}};$$

$$l_{ht} = \Xi_2^{\frac{1}{1-\gamma}} (z_1 z_2)^{\frac{1}{1-\gamma}}; \quad \frac{\partial l_{ht}}{\partial z_1} = \frac{\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma} z_1^{\frac{\gamma}{1-\gamma}} z_2^{\frac{1}{1-\gamma}}; \quad \frac{\partial l_{ht}}{\partial z_2} = \frac{\Xi_2^{\frac{1}{1-\gamma}}}{1-\gamma} z_2^{\frac{\gamma}{1-\gamma}} z_1^{\frac{1}{1-\gamma}}.$$

C.4 Numerical analysis

Substituting the numerical values in our calibration into the Jacobian matrix yields

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -0.01613 & -0.00082 \\ 0.09704 & -0.00260 \end{bmatrix} \longrightarrow \begin{bmatrix} -0.01613 & -0.00082 \\ 0 & -0.00756 \end{bmatrix}.$$

Therefore, the eigenvalues for this Jacobian matrix are -0.01614 and -0.00736, respectively. Given that the eigenvalues are real, distinct and negative, the model features a stability of saddle-path.

References

Howitt, P. (1999). Steady endogenous growth with population and r&d inputs growing. *Journal of Political Economy*, **107** (4), 715–730.

Segerstrom, P. S. (2000). The long-run growth effects of r&d subsidies. *Journal of Economic Growth*, **5** (3), 277–305.